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ABSTRACT

Students can learn to solve problems of qualitative integration and differentiation independently of their study of formal calculus or algebra. This exploratory study investigated the basic intuitions that elementary school children construct in their daily experience with physical and symbolic change. Elementary school children (n=18) were interviewed individually and in small groups to study how children use their knowledge about addition and subtraction to describe, predict, and explore situations of change. The paper is divided into five sections. Section 1 describes the authors' view on additive change and its relation to calculus. Section 2 presents a review of work related to the research on additive change. Section 3 uses the review to describe the goals of the study and the methodology utilized in the interviews. Section 4, the main component of the report, analyzes selected episodes from the interviews. Section 5 discusses the three themes that emerged from the study: (1) additive change: a web of relationships; (2) the many graphs of additive change; and (3) qualitative relations. The authors conclude that the interview episodes indicate that qualitative reasoning is not pre-quantitative or less sophisticated than quantitative judgments. (Contains 28 references.) (MDH)

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Contents

Preface

TERC is a nonprofit education research and development organization founded in 1965 and committed to improving science and mathematics learning and teaching. Our work includes research from both cognitive and sociocultural perspectives, creation of curriculum, technology innovation, and teacher development. Through our research we strive to deepen knowledge of how students and teachers construct their understanding of science and mathematics.

Much of the thinking and questioning that informs TERC research is eventually integrated in the curricula and technologies we create and in the development work we engage in collaboratively with teachers. In 1992 we launched the TERC Working Papers series to expand our reach to the community of researchers and educators in similar endeavors. The TERC Working Papers series consists of completed research, both published and unpublished, and work-in-progress in the learning and teaching of science and mathematics.

Introduction

As part of the Measuring and Modeling project, we began to study how high school students without background in calculus think about calculus problems set in physical contexts that students can experiment with and relate to their common knowledge. Through a series of teaching experiments, we started to recognize the richness, diversity, and complexity of students' approaches (Nemirovsky & Rubin, 1991; Rubin & Nemirovsky, 1991). We came to realize that students can learn to solve problems of qualitative integration and differentiation quite independently of their study of formal calculus or algebra. Rather than seeing calculus as an "end point" in mathematics education, we began to develop a long-term view of calculus learning: that an understanding of calculus develops from basic intuitions that children construct in their daily experience with physical and symbolic change. This question became more and more central: What are the strands of knowledge that enable us, from early childhood to adulthood, to deal with calculus situations?

Our efforts in this exploration moved us to work with elementary school students. We interviewed children individually and in small groups, and observed classroom activities after designing curricular materials. We report here children's work in individual interviews. Currently we distinguish several domains in learning calculus. This paper focuses on one of them: additive change. We are interested in studying how children use their knowledge about addition and subtraction to describe, predict, and explore situations of change, as well as how, in doing so, they learn about additive structures.

This paper is divided into five sections. Section 1 describes our current view on additive change and its relation to calculus. Section 2 presents a review of work related to our research on additive change. Section 3 uses the review to articulate what is specific about our work and introduces the interviews that are the core of this paper. Section 4, as the main component, analyzes selected episodes from the interviews. Section 5 discusses the themes emerging from this research.

The research that we report here is in an exploratory phase; to this extent, we are more interested in opening questions and issues than in being conclusive. We wish to articulate links and relations among areas of research that traditionally have been disconnected and to illuminate a conception of calculus as the mathematics of change.

Section 1: Our Views on Additive Change and Calculus

Number Sequences and Change

A number sequence is a useful mathematical tool to describe any situation of change, such as people coming in and going out of a place, water filling up a container, motion of objects, accumulation of coins in a piggy bank, and so forth. A number sequence may describe either continuous change (e.g., the height of a plant) or discrete change (e.g., number of people in a room). A number sequence for discrete change usually includes only whole numbers, whereas decimals can be part of a number sequence for continuous change. The following number sequence may describe the number of people in a room at different times: 2, 5, 7, 4, 3.

This number sequence expresses different quantifications of people in the room and it expresses them in order. We not only know that at a certain time there were seven people in the room and at another time there were two, but we also know that the two-person situation happened before the seven-person situation.

In any situation of change, a distinction can be made between quantities and change of quantities (e.g., number of people in a room and number of people going in and coming out of a room). They are mutually related; having some information about the former, we can infer information about the latter and vice versa. There are four fundamental properties about this relationship between number sequences for quantities and quantity-changes. These properties are particularly important to us because they relate to general ideas of calculus.

> (1) In going from quantities to quantity-changes we lose a piece of information. We are left with information about differences, but we lose the original values. In going from the quantity-changes to quantities we can compare quantities but not their individual values. For example, from the sequence of changes of people in the room, we may answer questions of the type "How many more or less people . . ." but not of the type "How many people . . ."
>
> (2) The role of the sign is different in the sequence of quantities and in the sequence of quantity-changes. The meaning of the sign in the sequence of quantities is highly contextual and conventional for each type of change. For the number of people in a room, for example, a negative number is senseless (unless you look for a cumbersome interpretation), whereas, for the position on a track, a negative number may mean that the object is at the left of a certain point along a certain line. This is not the case with the use of the sign in the sequence of changes when the sign serves the purpose of discriminating increases from decreases. This meaning of the sign in the number sequence of changes does not depend on the nature of the change itself.

(3) A similar distinction can be made regarding the use of zero. Zero in the sequence of changes means no change; that is, the quantities before and after the zero change are the same. This is a general meaning for any kind of change. On the other hand, a zero in the sequence of quantities is more conventional and idiosyncratic. A zero in position, for example, indicates a specific position that is given by our arbitrary definition of the origin point. A zero in volume or number-of-things in a container (such as number of people in a room) is usually taken to mean the condition of emptiness.

(4) The distinction between number sequences for quantities and number sequences for quantity-changes is relative. A sequence of quantity-changes can be taken as a sequence of quantities on the basis of which a new sequence of quantity-changes is generated, and vice versa.

Additive Change and Calculus

Additive change is the core idea of the discrete "side" of calculus. It was a guiding notion for Leibniz in his approach to the development of calculus (Edwards, 1979). In general, additive change enables us to construct discrete descriptions of continuous change. This general parallelism between additive change and discretization of change can be illustrated by showing how calculus theorems admit discrete versions that are based on additive change. Strang (1990) proposed the use of additive change to introduce many theorems in college calculus courses. Following are a few examples; we are interested in tracing their appearances as "theorems in action" (Vergnaud, 1990):

Fundamental theorem of calculus. The discrete correlate of the first fundamental theorem of calculus is that given a sequence of quantity-changes one can always construct many corresponding sequences of quantities whose successive differences will be identical to the given sequence. The central idea here is that the operation of a running sum is the opposite of taking successive differences. For example, say that this is a sequence of net people entering (positive) or leaving (negative) a room: +1, +2, +3, -2, -1. This theorem says that one can always find many sequences for the number of people in the room whose net differences will coincide with the given one, such as 25, 26, 28, 31, 29, 28 or 2, 3, 5, 8, 6, 5.

The theorem also states that all the sequences of quantities corresponding to a given sequence of quantity-changes differ only by a constant. For instance, the example sequences above differ, term by term (25-2, 26-3, 28-5, etc.), by the constant 23. This theorem can be conveniently expressed from a graphical point of view: if one graphs many possible sequences of quantities, for a fixed sequence of quantity-changes, all of them will be vertical translations of the same graphical shape.

Sign of the derivative. There are several calculus theorems dealing with extreme points. One closely related property in additive change is that a change of sign in the sequence of quantity-changes indicates a local extreme in the sequence of quantities. A change from + to - indicates a local maximum and a change of - to + corresponds to a local minimum. Associated ideas are the following: if the sequence of quantity-changes is positive, the sequence of quantities is strictly increasing; if the sequence of quantity-changes is negative, the sequence of quantities is strictly decreasing; if the sequence of quantity-changes is zero, the sequence of quantities is constant.

The Overall Picture

We are not saying that calculus is "easy"; we do not expect to produce another "Calculus Made Easy" textbook. What we are recognizing is that if calculus is identified with the mathematics of change, then its learning must grow out of the experience, reflection, and representation of change. This is similar to the fact that the activities of counting are fundamental for the construction of knowledge about natural numbers, or that the manipulation of objects is critical for the learning of geometry. In the same sense that we can acceptably say "a child who discovers that the numerosity of a set does not depend on how one counts learns something fundamental about numbers," we want to say that a child who recognizes how subtraction reverses addition learns something fundamental about calculus.

Another important reminder is that we conceive additive change as only one of the strands of knowledge involved in learning calculus. Other strands that we are studying are continuous variation and functions. We want to learn how additive change relates to other areas of calculus knowledge.

The mathematics of change may reach high levels of complexity. The domain of difference equations, which are expressions of additive change, is currently one of the most complicated and promising areas of research in mathematics, closely connected with chaos, fractals, and modeling. But we want to change the common image by which calculus is perceived as an end point, that comes after "pre-calculus" mathematics (only if the student passes an appropriate "calculus readiness" test). Calculus is, or should be, a natural extension of what we already know about change in the physical and symbolic worlds. In this approach learning calculus may become a long-term process that involves the nurturing of fundamental intuitions that children develop from an early age.

Section 2: Review of Literature on Additive Change

The fundamental operations that relate the sequence of quantities with the sequence of quantity-changes are addition and subtraction. Often the common notation for addition is used to express the sequences. For example,

$$2 + 3 + 2 - 3 - 1 = 3$$

can be interpreted through the former problem of people in the room: there were two people at the beginning, then three more came in, then two more came in, then three left, then one left, and three remained in the room. In this notation the first and last numbers stand for initial and final numbers of people in the room respectively; that is, they belong to the sequence of quantities, whereas all the other numbers belong to the sequence of quantity-changes. This number sentence does not include explicitly all the intermediate quantities, yet it gives us all the necessary information to compute them. The literature on additive change focuses on this type of number sentences for single-step additions or subtractions.

The Semantics of Addition and Subtraction

Additive change is usually identified as one of the possible meanings of number sentences. In other words, additive change has been studied as part of the semantics of addition and subtraction. How the semantics of addition and subtraction affects students' performance has been studied extensively by many researchers (Bergeron & Herscovics, 1990; Carpenter & Moser, 1982; De Corte & Verschaffel, 1987; Nesher, Greeno & Riley, 1982; Vergnaud, 1982). The patterns of difficulties shown by children vary according to the semantics of addition and subtraction. One variable that strongly affects the performance is what is the unknown. Problems of Change in which the unknown is the final quantity (called Change 1) are the easiest, but when the unknown is the initial quantity (Change 3) the frequency of mistakes is significantly higher. The wording of the problems and the order in which the information is presented also affect performance.

Researchers have developed several taxonomies to distinguish types of semantics (Nesher, Greeno & Riley, 1982), trying to elucidate why children show differences in their rate of success and failure for each semantic type. Included in those schemes are two semantic interpretations to addition that we want to compare: Combine and Change. A number sentence such as 3+4=7 may represent either kind of word problem such as:

Combine: Tom has three candies. Mary has four candies. How many candies have they altogether?

Change: Tom had three candies. Mary gave him four more candies. How many candies does Tom have now?

In a Combine situation the addition describes a part-part-whole relationship, whereas in a Change situation it describes a sequence of increases/decreases over time. The contrast between Combine and Change points out three aspects from the literature that are particularly relevant for our research:

Unary operations and the order of the changes. As Freudenthal (1983), Weaver (1982), and others have pointed out, addition and subtraction can be conceptualized by unary or binary operations. In Combine, addition tends to be thought of as a binary operation, whereas in Change it is more likely to be thought of as a unary operation. A number sentence like 3+4=7 is more likely to be interpreted in Combine as (3, 4) -> + -> (7), but in Change as (3) -> +4 -> (7). In other words, in Combine + is the operator acting on both numbers 3 and 4, whereas in Change +4 is an operator acting on 3. Viewing addition as unary or binary carries consequences, such as on the understanding of properties like commutativity and inverse. For example, in Combine 4+3 and 3+4 are likely to be considered two equivalent ways of putting Tom's and Mary's candies "together." However, for the Change problem 4+3 can be perceived as different from 3+4 because it may make a difference whether Mary gave four candies and Tom already had three or if she gave three and Tom already had four. To us, this is relevant because we want to explore children's ideas about how the order in the sequence of quantities and quantity-changes affects the results.

Number sequences as changes that happen "out there." Change alludes more directly to something that happens "out there," whereas Combine often expresses just a change of viewpoint in the observer. For example, one can be told that three people are in a room and four are in another room and one can imagine how many people there are "altogether" without imagining any actual change in the distribution of people in the rooms.

The order of a number sequence as a temporal order. Since Change emphasizes the sequence of events over time (whereas Combine emphasizes the non-temporal relationships part-part-whole), a Change interpretation of a number sequence is closely connected to the identification of the order in the sequence with a temporal order.

Counting, Conservation, and Addition

Another strand of research that is relevant for our work focuses on qualitative judgments of "more," "less," and "same" after additive change occurs. This work grew in reaction to Piaget's (1952) seminal study on children's concept of number. Piaget postulated that number conservation was constructed after the child conceived the reversibility between addition and subtraction. He also considered that the landmarks in development were logical structures, such as one-to-one correspondence, and he dismissed the role of procedural abilities, such as counting. A stream of research has flowed since, striving to understand the relationship between conservation and addition (Brush, 1978; Goth, 1980; Smedslund, 1966;

Starkey, 1978; Starkey & Gelman, 1982) and between counting and addition (Bergeron & Herscovics, 1990; Fuson, 1988; Gelman & Gallistel, 1978; Ginsburg, 1982; Steffe & Cobb, 1988). From this literature we want to highlight two aspects:

Counting/computing and numerical reasoning. This is a distinction noticed by several researchers (Bergeron & Herscovics, 1990; Gelman & Gallistel, 1978). Numerical reasoning refers to the use of number relations. Depending on the situation, children solve situations of addition by counting or by using number relations. For example, 4+3 can be solved by counting objects or by inferring that it has to be one less than eight because 4+4=8. This is important for our work: we have observed how children shift from one perspective to the other in their analysis of sequences of changes.

Counting/computing and judgments of more/less/same. The interplay between these two kinds of activities is central in our analysis. The literature, however, is restricted to a developmental analysis of basic discriminations of more/less/same, given one or two changes of one or two units. Through the interviews we became aware of how complex and rich children's reasoning can be for the determinations of more/less/same in the context of a sequence of changes.

Addition and Notations

Other related areas of research focus on notations for additive structures (Vergnaud, 1982). The use of conventional symbolisms often conceals ambiguities or collapses multiple meanings. In additive change we have to distinguish at least three meanings for the sign:* indicating locations or states (e.g., left or right of zero); operations, moves, or shifts (e.g., decreasing/increasing); and reversals, opposites, inversions, or negation (e.g., opposite/same). A single number sentence for a setting such as a number line may include the three meanings; for example, in -6-(-2) the first minus indicates location, the second reversal, and the third move. Thompson and Dreyfus (1988) explored how children dealt with these different meanings by using a computer-based microworld.

Many curriculum developers make explicit the distinction between the sign as a location or as an operation. The Comprehensive School Mathematics Program (CSMP), for example, uses ^, like ^1, to denote left location and -, like -1, to indicate move to the left. For example, ^2-1 would mean "start on 2 units to the left of zero on the number line and move one unit further to the left." The distinction location/move is more meaningful to children than the use of the sign as an inversion. Ball (1990) wrote a vivid account of her experience teaching negative numbers to third grade students. She tried different settings (money, an elevator moving up and down, etc.) and observed with frustration the difficulties the students experienced as

* We are currently not using the electric-charge setting in which there are two kinds of amounts that annihilate each other (Battista, 1983).

they attempted to construct a meaning in-the-setting that was mathematically consistent for expressions such as 6-(-3)=9.

A difficulty of understanding the minus sign as a reversal stems from the fact that using - as inverse, or + as direct, does not correspond to a specific action in-the-setting. It is an action on actions; its referent is in the notational context, for instance, in this expression -(+3-2) the - (. . .) does not indicate something to do on a track or a number line, instead, it is a transformation from +3-2->-3+2 before the actions on the track take place. There are situations in which some students come to the idea of reversing the operations (Pesci, 1990). In posing "backwards" problems such as ?+3-2=4, we have observed students developing the intuition that if one starts with four and does all the changes in reverse, one gets the initial location. This insight may make meaningful the use of a symbol for reversal.

This literature connects to our interest both in encouraging children to express themselves by creating their own notations and in introducing conventional systems in the context of children's invented symbolisms.

Section 3: Our Preliminary Work in Additive Change

What Is Different in Our Work?

The specificity of our research orientation in additive change is conveyed by the following:

> We are particularly interested in how children deal with additive change using numerical reasoning, because the fundamental ideas of the mathematics of change can be expressed through these relationships. We work with small numbers and *sequences* of changes, focusing on how students construct relationships between sequences of quantities and sequences of quantity-changes.
>
> Qualitative reasoning is at the core of our inquiry. We design situations highlighting relations of more/less/same and orders of magnitude (e.g., small, medium, large numbers).
>
> We are interested in children's notations and graphing. To our knowledge, the graphing of additive change is a specific subject matter that has not been studied.
>
> We are exploring additive change as part of the knowledge necessary to the interpretation of calculus problems.

We think that this research orientation is critical not only to illuminate how calculus, viewed as the mathematics of change, can be introduced across all the educational levels, but also to clarify how this mathematics has the potential to enrich the teaching and learning of addition, graphs, and number sequences.

The Interviews

We began a study of how children work with situations of additive change in individual interviews, small group interactions, and classroom activities. The research presented here focuses on the individual interviews we conducted with 18 students in the summer of 1991.

Through the design of situations for children to learn about additive change, we made a distinction between Amounts and Positions. Examples of Amounts are the number of people in a room, cars in a parking lot, letters in a book, the volume of water in a container, and the height of a plant. Examples of Positions are the locations of a person in a street, the water levels in a container according to a certain scale, and the floors at which an elevator may stop. Obviously, these two types of change are closely related and can correspond to each other. For example, the volume of water in a container (an Amount) can correspond to the water level (a Position). There are some general differences, however. For example, negative values tend to be more easily recognized as appropriate for a Position than for an Amount. In an Amount, the zero tends to be identified with the condition of emptiness, whereas in a Position the zero is more conventionally decided. In an Amount, there is a "natural" direction of increase, whereas in many Position contexts the direction of increase is readily recognized as conventional (e.g., why should "more" be identified with "more to the right" in a scale?).

The interviews that we report in the next section focus on an Amount, the number of blocks in a bag. The changes are presented as "putting ins" and "taking outs" of blocks to and from the bag. We try to grasp how students begin to distinguish different meanings. Frequently, these emerging distinctions are expressed in the invention of idiosyncratic symbols. Examples of these are the spontaneous use of marks to discriminate between ins and outs of blocks in the bag.

The sequence of situations varied from child to child because the interviewer strived to follow the students' ideas and because, as we learned from observing the children, we redesigned the problems and the ways of posing them. We used paper bags and plastic blocks, and supplied the students with big sheets of paper and markers. The interviews were videotaped.

We can classify the problems we posed into three broad types:

Type 1: Interplay of changes and accumulations. Given partial information about the sequences of quantities or quantity-changes or both, the interviewer asked the child to think about the missing information.

Type 2: Net change. These problems generated questions such as whether the order of the changes affects the net change, what happens to the net change if all the signs of the changes are reversed, what happens to the ending number if you change the starting number.

Type 3: Directions of change. These problems were based on sequences of signs such as + + + + - - - +, and the student was asked to think about maxima, minima, and relations of more/less along the sequence of accumulations.

In presenting this research, we have chosen the six episodes that follow because the dialogue between the interviewer and the child is clear and articulate and because the occurrences illustrate the broad range of approaches manifested by the children during the interviews.

Section 4: Selected Episodes

Number Relations and Net Change

The interviewer, Mark, explored sequences of changes and net change with Chris and Jay during the summer before their first grade year. The work of these two boys created episodes that complement each other. Chris is solving problems in which the initial number and final number of blocks in the bag have been given; he is asked to think about possible sequences of changes. Jay, on the other hand, in addition to work with possible sequences of changes, is trying to solve the initial number and final number of blocks from a given sequence of changes.

Chris (age 6; 14-minute episode)

Segment 1: 4 to 7 in two and four changes

While working with a previous problem, Chris invents a way of distinguishing the "taking outs" from the "putting ins" by marking the numbers with "t" and "p." Here, for instance, he keeps track of the sequence start with five, put in four, take out two and take out one with this expression (see Attachment 1, page 51):

$$p_5\, p_4\, {}^{t}{}_2\, {}^{t}{}_1.$$

Mark: There are four in there now, OK [Chris writes a 4 beneath a *p*]. And here's what I want you to think about. Can you make two changes to the four that I started out with so that I would end up with seven in the bag? Two changes.

Chris: [after 10 seconds] Put in one and then put in two.

Mark: Put in one and then put in two, and then I should end up with seven?

Chris: Yeah.

Mark: How do you know?

Chris: Because you know that there's three more numbers from four to seven.

Mark acts out the two changes proposed by Chris, and Chris counts the final number of blocks verifying that it is seven.

Mark: Bingo! Let's make it a little bit more of a challenge. We'll put four in to start, same as before. Now can you start with four and get to seven, using four changes?

Chris: [smile] Ah, zero, one, one, and one.

Mark: Now, why do you think that one is going to work? It didn't take you too long to figure out. Explain that to me.

Chris: Because zero . . . and three ones . . . is three.

Mark: . . . Now a bigger challenge since you're getting so good at these. Can you go from four and seven, again with four changes, but not using a zero as any of your changes?

Chris: [after 20 second] Um, OK. Two and one and two zeros.

Mark: Ah, two zeros.

Chris: You told me not to do *a* zero. [laughs]

Mark: . . . Can you go from four, to start with, seven to finish, with four changes, not using any zeros as your changes? In other words, not have any of the changes be zero?. . .

Chris: Impossible.

Mark: Impossible? Why?

Chris: Because it's only three numbers to seven from four. So you can't do that without any zeros.

Mark: Hmm, well . . . so you think without zeros you can't do it? Suppose I got you started. I have to say that maybe you can. OK. We start with four [writes 4] and we want to end up with seven [writes 7], right? Suppose I gave you one of the four changes, and then you only had to come up with three of them. Let's try that, OK. All right, the first change, suppose we start with this [writes +2]. No, wait, you weren't using that [changes the +2 for a P2]. You were using put, right. So for the first change suppose I told you one of the four changes was put in two. And you have to come up with three other ones. Do you think you can do it?

Chris: [after 30 seconds] A half . . . OK, then a half and then a half, and then one . . .

Analysis

In this segment, Chris faces the problem of going from 4 to 7 in more than three changes. A general observation is that Chris does not have difficulties distinguishing changes from number of blocks in the bag. He creates a notational way of distinguishing changes from accumulated number of blocks: changes are numbers with a "p" or a "t" above. Changes have the specific attribute of being "put ins" or "take outs." In other words, signed numbers denote changes and unsigned numbers quantify the number of blocks in the bag. This is a very common approach that we observed in our interviews. A change is perceived as an action, with two possible directions, whereas the accumulated numbers are quantifications of static sets. The distinction is salient, and children working with the setting of the ins and outs of the bag show fluency in keeping track of the changes separately.

In trying to solve the problem to go from 4 to 7 in four changes, Chris assumes tacitly that the changes have to be positive. The constraint is perhaps induced by his viewing the inquiry as a "going up" problem. His first solution to the conflict of having to make positive changes, when the final number of blocks has been reached, is to add zeros as changes. By conceptualizing zero change, Chris applies an idea that is mathematical in nature. There is nothing in the physical realm that pushes us to consider zero change. It can be omitted altogether. Zero change makes sense only as part of a mathematical system that creates a generic "space" for changes. Mark poses such a space when he asks for "four changes." This mathematical framework elicits the need to think of "no change" as a change.

After Mark repeats the request of not using zeros, Chris says that it is "impossible." What kind of knowledge is behind his conviction that this problem cannot be solved? Chris does not come to this conclusion by considering all the possible changes, counting the final number of blocks and then seeing that in no case he gets seven as the final number. He knows that in order to accumulate three blocks, always putting in more, he cannot use more than three steps. This is a type of knowledge that we want to distinguish from counting-based knowledge. It is related but independent from counting. Counting could be a strategy to try possibilities, but not the source of an assertive "impossible." Following other researchers (Bergeron & Herscovics, 1990; Gelman & Gallistel, 1978), we want to make the distinction between number abstraction (determination of numerosities, counting, etc.) and number relations (more, less, derived number facts, etc.). We would like to identify Chris's recognition of impossibility as stemming from his knowledge of number relations: the more positive changes to get a certain number, the less in each change; positive changes increase the number of blocks; the number of blocks change by units of one, etc.

Chris reveals an interesting twist of logic to overcome the impossibility of a solution after Mark insists on his not using zeros. The more positive changes to get a certain number, the less in each change; but how could the changes be less than one? In breaking through this puzzle, and taking 30 seconds to think about it, Chris goes beyond the physical boundaries of real blocks. He imagines fractional changes. We believe that such venturing beyond the limits of physical elements by an act of imagination configure the terrain where mathematical ideas thrive.

In suggesting a first change, Mark introduces the conventional symbol of "+" for a put in. Although he immediately converts it to Chris's symbolism with a "p," this allusion is enough for Chris to replace his notation with the conventional one. We see this in the next segment.

Segment 2: 6 to 4 in four changes (a)

Mark: So you're using fractions, hmm. Let me give you another problem sort of like this one, OK. Let's say we start out with six in the bag [puts six blocks in the bag]. OK, this is a totally new problem . . . I have six in here. Can you get me down to four using three changes?

Chris: Hmmm. There's one half then um, oh yeah, OK, yeah. [after 25 seconds] There's one. OK, there's one and then two and then three, and then take away one.

Mark: OK, how many changes is that?

Chris: Four.

Mark: So one and two and three. . .

Chris: And take away one.

Mark: So you said, put in one, and two and three?

Chris: Yeah, and then take away one.

Mark: And take away one. Let's see, write that down. Now what did we start with?

Chris: OK, we start with . . . [writes 1+1+1] hmmm . . . What's the take away sign?

Mark: Um, it looks like this. [Mark writes -.]

Chris: Um, I think we want this [writes -1].

Mark: And that's going to give us what?

Chris: Four.

Mark: If we start with what?

Chris: No. If you start with four and then you go to six.

Mark: That will get us from four to six?

Chris: Yeah.

Analysis

Let us analyze in detail the dialogue after Mark poses the new problem. Chris's first impulse is to use halves again ("There's one half . . ."), but as he tries to figure out the sequence of changes he has an insight ("oh yeah, OK, yeah"): he is now aware that the changes can be decreases too. Possibly, this insight is triggered by Mark's posing a "going down" problem. Immediately Chris goes back to the integers ("there's one") and creates a sequence that ends with "and then take away one."

Their subsequent conversation is a clarification of Chris's initial thought. Chris transforms the problem as he appropriates it: the problem becomes going up from 4 to 6 in four changes. By "there's one and then two and then three" he means 1+1+1. Note how Chris goes back and forth between registering the accumulated numbers ("there's one and then two and then three") and the changes (1+1+1).

Segment 3: 6 to 4 in four changes (b)

Mark: OK, you're right, it would. Now what if we started with six and went to four . . .

Chris: How many changes?

Mark: Four changes. Same number of changes.

Chris: OK, four and six. Four and six. OK, now you take away one [writes -1] . . . no, [scratching out -1, then writing 1+1-4] There. There's one plus one, take away four.

Mark: What is that going to give me?

Chris: Four, from six.

Mark: If I start with six that will get me down to four?

Chris: Yeah.

Mark: Explain that. Explain how that works.

Chris: It goes, I do two and then . . .

Mark: What do you mean do two?

Chris: . . . You put in two numbers, so that's one plus one, uh-oh . . . I did only three changes [scratching out].

Mark: Ah, well, can you change it so that it works with four?

Chris: Now. . . . That's what you do [writes 1+1-2-2]. And that's . . .

Mark: You know what, this looks sort of like what you did here [pointing to 1+1-4], where you had one plus one, minus four, take away four.

Chris: But that's . . . that's four changes [pointing to 1+1-2-2]. And that's [pointing to +1+1-4] only three changes.

Mark: I see. So how did you get this one [1+1-2-2] to be right here?

Chris: Because I only . . . I changed . . . the two plus two and that was four down here [in the former three changes] . . .

Mark: Two and two. [pointing to the -2-2] It's not exactly the same as two plus two is it? What's different about this one?

Chris: But . . . because I mean it is take away.

Mark: I got you. OK, so if we start with six, these changes will give us four? Let me just check that. So six in here so far. If I put in one. Put in one. [put one block and another in the bag]

Chris: Then take away four.

Mark: Take away four?

Chris: Take away two and then take away two more.

Mark: Which is the same as taking away four? [takes away four]

Chris: Yeah. So you've got four? Four.

Mark: . . . If I wanted to do what we just did in one change, how would I have done that?

Chris: You would take away two.

Analysis

In this segment Chris coordinates completely the increasing with the decreasing changes as two symmetric possibilities. At the same time, he approaches the problem by composing and decomposing numbers:

—"I do two and then . . . so that's one plus one" (2 -> 1+1)

—"the two plus two and that was four down" (4 -> 2+2)

—"because I mean it is take away" (-4 -> -2-2)

—"You would take away two" (1+1-2-2 -> -2).

Chris talks about the -2-2 as "two plus two" as if he considers the quantity 4 separately from the sign, which is always negative because "I mean, it is take away." We identify composing and decomposing small numbers as critical areas of knowledge on number relations (Marton & Neuman, 1990). At the end of the segment Chris expresses the net change of the problem (-2). Viewed in the context of the whole segment, we want to highlight how the notion of net change is intimately linked to number relations. The interplay between net change and sequences of changes elicits in Chris two interconnected ideas:

—One can increase the number of changes, for the same net change, by decomposition.

—One can decrease the number of changes, for the same net change, by composition.

Jay (age 6; 25-minute episode)

Segment 1: 4 to 7 in four changes

Mark: There's four [blocks] in here [the bag]. Can you think of a way to make there be seven using four changes?

Jay: Impossible.

Mark: Why do you think it's impossible?

Jay: Because if you add up it's impossible. But taking two away then adding four and that would make seven.

Mark: And how many changes would that be?

Jay: Four.

Mark: . . . I'm going to write over here [writes 4]. If we started with four can you show me what those four changes would be so that I would end up with seven?

Jay writes 4-1 4-1 + 2 + 2 (see Attachment 2, page 52).

Mark: ... Explain to me what you just did.

Jay: I just did four minus one, then another four minus one. Then add two more, and then another two more and that means you took away. [after silent time pointing to the two 4-1's] All right. Four minus one equals three. So do another four minus one equals three [pointing to the second 4-1]. That gives six. Hmmm. That made it a little hard.

Mark asks Jay to act out the sequence with the blocks and the bag. Jay executes the changes of his number sentence; he starts with four blocks in the bag, then he takes out one, another one, then he adds two and another two. Before Jay counts the final number of blocks, Mark asks:

Mark: How many do you think that's going to be?

Jay: Seven. [he counts] One, two, three, four, five, six.

Mark: ... This might help you out. You did four different things here, four different changes [pointing to the number sentence]. Could you do one of those changes a little differently so that it would ... work for you?

Jay: Yeah.

Jay writes this new number sentence 4-1 4-1 +2 +3 (see Attachment 2, page 52). Then he starts to act out the corresponding sequence of changes. First Jay puts four blocks in the bag, then he takes out two.

Mark: Why did you take out two? Because I see a four minus one and a four minus one [on the second number sentence].

Jay: Because take away one in each of these [pointing to 4-1 4-1].

Mark: It's just like taking out two?

Jay: Yeah.

Mark: Oh, OK.

Jay makes a mistake in following the second number sentence, putting in twice two blocks and then three blocks. After counting and seeing that he has "too many," Mark explains to Jay that he has done the +2 twice.

Mark: I'm curious to know why did you change +2+2 to +2+3 [pointing to both number sentences]?

Jay: Well, all this [points to the first number sentence] was six, so I put in one more, to make seven.

Analysis

Initially, Jay says "impossible" to the problem of going from 4 to 7 in four changes. It is the same reaction that Chris has to this problem (see Segment 1, page 10). But almost immediately, Jay realizes that it is impossible if you only "add up." Jay proposes a solution including taking away (taking two away then adding four). The way Jay utters the changes seems like a two-changes solution, but he promptly says that it consists of four changes. As his first number sentence makes clear, he means taking away two in two steps and adding four in two steps. Already he has grouped the changes in taking aways and putting ins. We observe children constantly composing and decomposing changes (see Segment 3, page 14), very often saying something like "take away two" but meaning, through their notations and actions, "take away one and then another one." This often leads Mark to interpret that the child has changed the problem. Another example of this occurs midway in their conversation.

Jay's first number sentence is a sequence of four pieces: 4-1/4-1/+2/+2. Each piece corresponds to a particular action of change that is part of a temporal sequence. The second 4-1 is independent from the first 4-1; they have the same form because they reflect the same action, take away one. This is clear in Jay's language: "I just did four minus one, then another four minus one." The notation for the action of taking away includes an initial number of blocks. This is not the case with addition. Jay does not write, for example, 4+2 for putting in two blocks. This may be part of a more general asymmetry in the way Jay and other children distinguish taking aways and putting ins. To put in blocks in the bag one uses the actual blocks from an unquantified pile of blocks on the outside, whereas to take away blocks from the bag one can do so only if an appropriate initial number of blocks are already in the bag.

After writing his first number sentence, Jay gives us an example of how the use of notations may shift the child's thinking in unexpected ways. The first number sentence reflects a sequence of four actions of change that he imagines by adding and subtracting blocks in the bag. But, after a time, the "four minus one" means "three." So 4-1 4-1 "gives six," which he thinks is impossible, because one cannot take six out of four.

By acting out the number sentence, Jay confirms that he needs one more block, so he copies the first three components of the number sentence and changes the +2 to the +3. Jay explains this process: "Well, all this [points to the number sentence] was six, so I put in one more, to make seven." Jay's statement "all this" reflects his perception of the number sentence as a whole that "is" the final number of blocks.

Segment 2: 5 to 6 in two changes

Mark: Starting with five do you think you can make it come out to six with two changes now?

Jay: Mmm-hmm . . . I will do this one [pointing to the first number sentence]. You said this is five [pointing to the first 4]. Let's say these two [the two 4's in 4-1 4-1] are five. Five minus one, and another five minus one, and plus two and plus two, I think would make six.

Mark: Why do you think so?

Jay:	Because it's [pointing to first number sentence as a whole] two more than were in the bag.
Mark:	In here [the bag] is how many?
Jay:	Five.
Mark:	And what's two more than five? I just didn't get what you meant when you said it's two more.
Jay:	Actually [silent time . . . I think I had it wrong. That's a little difficult.

Mark asks Jay to write down the numbers; Jay writes 5. Mark moves to write down a first change, but before he could do it, Jay says:

Jay:	Oh, yea, it would be a minus one, and then another plus one.
Mark:	Oh, you think that would do it?
Jay:	Well, no, minus one, [Marks completes 5-1] and then do a plus two [Mark completes 5-1+2=] and then it would make a six.
Mark:	. . . Why do you think that's going to work?
Jay:	Because five minus one equals four [pointing to 5-1]. And then mmm well, put this one [pointing to the 2]. Five and one back and it will make the five again, but we use a two to make six.

Then Jay acts out the changes of the third number sentence, and he verifies that he has six blocks in the bag.

Analysis

As Mark poses the problem of going from 5 to 6 in two changes, Jay makes a remarkable attempt to use the former notation to work out a new problem. Jay takes his first number sentence as a whole that "is" six. So in order to get from 5 to 6, he suspects that it will work if you start the sequence from 5, that is, changing the 4's of 4-1 4-1 into 5's: "*Let's say* these two [the two 4's in 4-1 4-1] are five. Five minus one, and another five minus one, and plus two and plus two, *I think* would make six." Then Jay looks at his first number sentence from another angle, now instead of "being" six, it produces two more blocks in the bag than there were at the beginning. This point of view, as well as the subsequent actual computation of 5-1 5-1 +2 +2, moves Jay to conclude "I think I had it wrong."

At the end of this segment, Jay takes on another approach. He does not focus on changing a number sentence anymore. Jay imagines a sequence of two changes, the first one decreases the number of blocks by one. The second change includes two components, going back to the original quantity of five and adding up to the final number of blocks: "Five and one back and it will make the five again, but we use a two to make six." This process of moving away and recovering the initial amount of blocks is a common strategy to increase the number of changes.

Segment 3: Adding 1 and 4 and taking out 3

Mark: This one, don't look for a second, OK. There's a little bit of mystery in this one. [Mark puts three blocks in each of two bags.] Now, OK, you can look. Now what I'm going to do is I have two bags here, and each of them has the same number now. I'm not going to tell you what it is.

Jay: OK.

Mark: But I'm going to tell you that there's the same number in each of these two bags. And now I'm going to make some changes to this bag and tell you how many you'd end up with. OK, so watch what I do.

Mark shows Jay four blocks and puts them in the bag.

Jay: Four.

Now Mark shows Jay one block and puts it in the bag.

Jay: Five. Including that one. You put in four, and you put in one, and that makes five altogether.

Mark: Do you want to keep track of this on paper, or are you just going to keep watching me.

Jay: I'm going to keep watching.

Mark: OK. If it helps you to try it on the paper you can do that too.

Jay: All right.

Mark: It's up to you.

Jay: I'll watch what you do.

Mark: So this is the next change.

Mark takes out three blocks.

Jay: Three.

Mark: I'll just take those out. OK. Now I can tell you what I have in here [in the bag where the changes took place]. But the first question I want to ask you is do you think that it's more or less than I started with?

Jay: Mmm, less.

Mark: Less?

Jay: Un-huh.

Mark: Why do you think it's less? I didn't tell you what I started with, right?

Jay: Actually, no, because you were putting those in five minus three makes, two of those in that, you put here. And I think it's going to be more than in that bag [the untouched one].

Mark: How many more do you think it's going to be? Do you have any idea how many more [are] in here than there?

Jay:	No.
Mark:	Do you think if I told you there were seven in there [in the changed bag], I'm not saying there are, but suppose I told you that, how many do you think about would be in that [the untouched bag]? You said this one [the changed bag] is going to be more than that [the untouched bag].
Jay:	Five [gesturing uncertainty]. I don't know.
Mark:	About five?
Jay:	Yeah.
Mark:	Do you think if I had seven here [the changed bag] I could have one in that one [the untouched bag]?
Jay:	No.
Mark:	Why not?
Jay:	Because you said they were both . . . the same amount in each bag. And you added in five and you took out three. That . . . so that means two of those that you put in, they're in there, so it's . . . it would be five . . . you said that there must be seven in here [the changed bag]. And there would be five in there [the untouched bag], and this [the changed bag], it would be two more in that [with emphasis] . . . So it would be seven in here [the changed bag].
Mark:	You feel pretty strongly about this it seems. Let's just see. [dumping out all the blocks from the changed bag] OK, so there really are . . .
Jay:	One, two, three, four, five.
Mark:	Do you have any prediction about how many we're going to have in here [the untouched bag]? What you had at the beginning?
Jay:	We're going to have three in there [very assertive].
Mark:	Three?
Jay:	Yeah.
Mark:	Let's see. [dumping out all the blocks from the untouched bag]
Jay:	I was right.

Analysis

Jay shows that he is keeping track of the individual and the accumulated changes. After Mark asks him about the possibility of keeping track on paper, Jay is very clear about his preference: "I'm going to keep watching." In Segment 1, Jay used written methods to annotate the changes but now he feels that they would not be helpful. This may be due to difficulties that he has encountered in using number sentences. After these experiences Jay is aware that by writing down symbols he thinks about the problems differently and not necessarily better.

After making the changes, Mark asks for a comparison (more/less) between the initial and the final amounts. Jay's first reaction is "less." Perhaps this is due to the last change being a taking out. But then he changes his mind. Note that Jay reconstructs all the changes by imagining specific blocks: ". . . because you were putting in *those* five minus three makes, *two of those* in that, you put here." There are more in the changed bag because some of those that were added remain in the bag. Although Jay says that two of those added will remain in the bag, he answers "no" when Mark asks whether he knows how many more blocks will be in the changed bag. One possible interpretation is that the question of "how many more" elicits from Jay the need to compare two unknown amounts, a task that he finds impossible.

As Mark suggests hypothetical amounts, seven and one for the final and initial amounts respectively, Jay has an easier time using what he knows about the changes. He becomes very articulate. He talks clearly about blocks as if they can be pointed out: "two of those that you put in, they're in there." In showing that seven and one are impossible, he constructs a condition of possibility given by the net change: "it would be two more in that [the changed bag]." This is an example of how the construction of the net change may shift the focus from the sequence of changes to a global number relation between start and final quantities, such as Jay's "two more."

In the second half of the dialogue, Jay is solving inverse problems (figuring out the initial amount knowing the changes and the final one). These are the most difficult change problems. Since many older students have difficulties understanding inverse problems, we bring attention to Jay's way of making sense of the situation. Jay pictures the net change by accumulating the changes in a very tangible mode; namely, by imagining pointable blocks to arrive at a concise expression of net change: two more here than there. Once he is fluent with the "two more" idea, whether this is a direct or inverse problem does not make much of a difference.

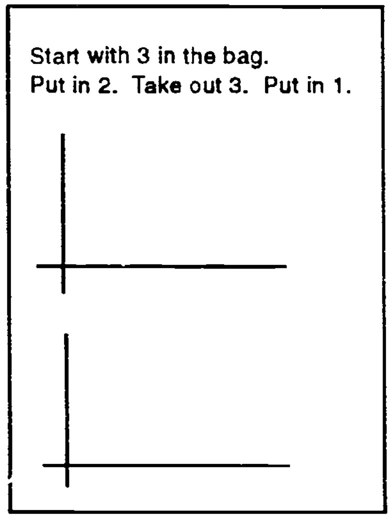
Graphs of Additive Change

Now we turn to episodes with Alice, who had completed fifth grade, and Rose, who had completed third grade. In these episodes, Mark, the interviewer, asks the two girls to create graphs representing a story of ins and outs of blocks in the bag: Start with three in the bag. Put in two. Take out three. Put in one. Their approaches are very different.

Alice (age 11; 11-minute episode)

Segment 1: Graphing 3 + 2 - 3 + 1

Mark shows Alice the following sheet that describes a sequence of changes (3+2-3+1) and separates the space for two graphs.



Mark: Have you made graphs in school?

Alice: Yes.

Mark: Do you remember what kind?

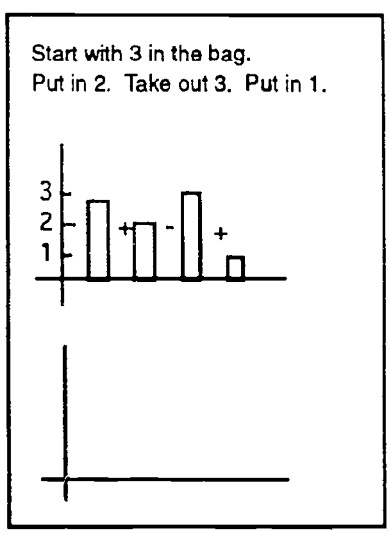
Alice: We've made circle graphs, line, like line graphs and a bar graph, I think.

Mark: So a few different kinds.

Alice: Yeah.

Mark: OK. Well, I would like you to try and think of a way of making a graph that shows these changes, what's happening in this bag . . .

Alice draws the following graph:



Graph 1

Mark:	Hmm, explain that one to me now.
Alice:	Well, you start with three ... And then you add some ... there's a bar that goes up to two. And then you take away three, and there's the bar that goes up to three. And then you add one ...
Mark:	OK, now can you tell from this graph how much you end up with in the bag?
Alice:	Well, you'd have to solve it, but you could ...
Mark:	So it doesn't show you itself what's in the bag at the end?
Alice:	Yeah ...

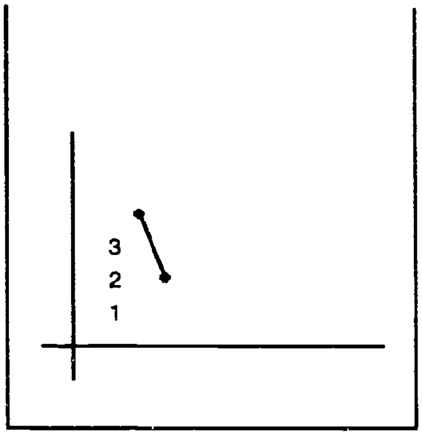
Analysis

Alice approaches the creation of a graphical representation trying to preserve the given story ("Start with 3 in the bag. Put in 2. Take out 3. Put in 1.") as the salient information to be conveyed. In other words, fidelity to the story frames her way of graphing. Her graph is a reproduction of the story in which the horizontal axis is used as a timeline. In her graph, she records three as the starting number then uses the vertical columns to represent changes. Changes are composed by an "amount" represented with the height of a column and direction denoted by the +/- sign. The point that we highlight is that her Graph 1 is a visual transcription of the story of changes (see Attachment 3, page 53). Even the positioning of the +/- sign reflects the order of the text (e.g., "put in" comes before "2" and so "+" comes before the corresponding column).

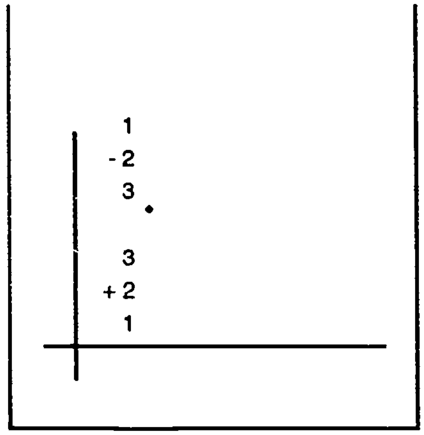
Segment 2

Mark:	All right, here's an idea then, could you show me on this [the bottom half of the sheet] what's in the bag at each of those changes?
Alice:	The same kind of graph?
Mark:	Well, does this graph right here show you how much is in the bag altogether at this point [pointing to a generic location]?
Alice:	No.
Mark:	That's what I'd like to see.
Alice:	A different kind of graph? OK.
Mark:	It might be like that graph but I want it to show me something a little different, which is how much you have in the bag at each of those changes.
Alice:	OK ...

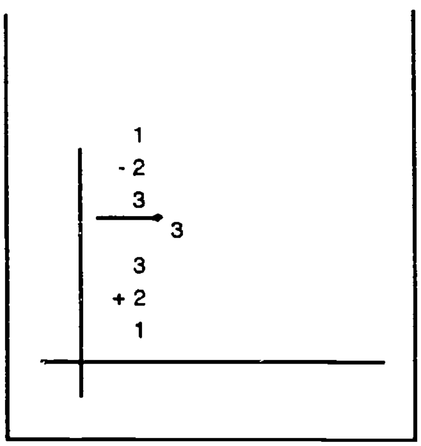
Alice draws the following graph sequence:



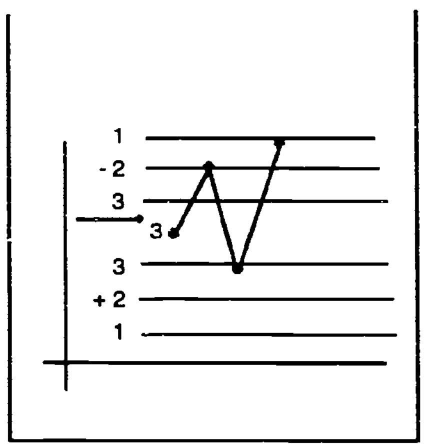
Graph 2a



Graph 2b



Graph 2c



Graph 2d

In Graph 2a, Alice traces a segment representing the "put in two" but immediately she erases it. She adds the upper 3 2 1, the signs, and the horizontal lines. Finally she draws the segments joining points.

Mark: You seem like you want to change something.

Alice: Yeah, it doesn't show you what it equals [the number of blocks in the bag].

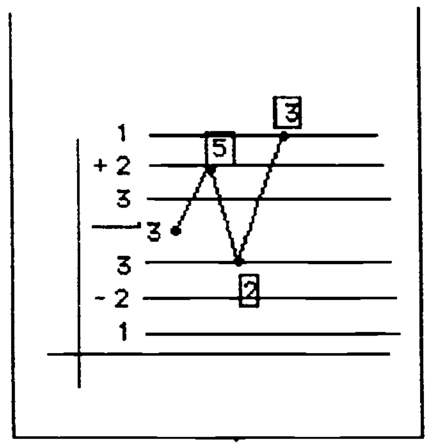
Mark: What's it showing?

Alice: It's just showing the problem like this one [the former graph in the upper half].

Mark: Oh, so you're showing the same information, you're saying?

Alice: Yeah . . . I didn't add it.

Mark asks Alice about the correspondence between the two graphs. Alice does not have difficulties in linking each vertex in the bottom graph with the corresponding column in the upper graph. Then Mark suggests Alice redo the graph in order to show the number of blocks in the bag at each time. Alice erases the broken lines, thinks, redraws the same lines, and adds the boxes with numbers in them to designate accumulation. This is the new graph:



Graph 3

Alice explains to Mark the meaning of each number inside the little boxes. Mark comments that the three looks bigger than the five.

Mark: All right. Well, what if I did this? What if I go ahead and add a change to this. What if I did . . . [adds a 2-unit column with - written above it, in the upper graph]

Alice: That's minus two. So you'd make a line that goes down to two, minus two, I mean. You'd make a dot and you'd write one. [See Graph 4, page 26.]

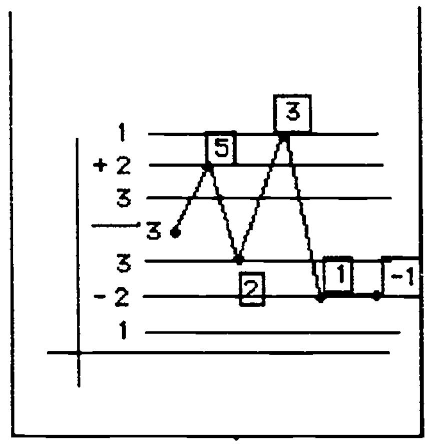
Mark: OK. What if I did minus two again? What would you show me down here?

Alice: You . . . you'd make a line just across like that.

Mark: And what would go in the box?

Alice: Mmm, negative one?

This is the new graph:



Graph 4

Analysis

Reacting to Mark's request to show the number of blocks in the bag more directly, Alice devises a new graphical structure through the process of successive accommodations. Let us follow it:

> In Graph 2a, Alice separates the amounts of blocks in the bag from the changes. Now there is a vertical column just for changes. She marks an initial point for the initial number of blocks and goes down to the "2" to indicate the "put in two," but as she tries to represent the next change (take out three), she realizes that she has no way to distinguish the put ins and take outs. She erases the segment.
>
> In Graph 2b, Alice adds an upper area for take outs and marks the two regions with a "+" and "-." In this way she solves the former problem of discriminating the two kinds of changes. This is an example of what we call "repair." Alice wants to incorporate a new piece of information (take out three) that she cannot fit in the structure, so she revises the structure as a whole.
>
> In Graph 2c, Alice makes more salient the initial amount of blocks as separate from the changes. She also changes the position of the signs; now positive changes are in the upper region.

After projecting this vertical structure onto the plane, by tracing all the horizontal lines, Alice is ready to represent the sequence of changes. But then she feels trapped by the structure that she has created because it does not offer a natural way to show the number of blocks in the bag.

Alice's Graph 2 incorporates more systemic elements: the changes and initial number of blocks are positioned differently, the positive and negative changes receive their own space in the graph, the events are now connected and highlight overall shapes, and so on. However, she does not make a generic graph; she is limited to the range of changes given in the story. To represent the action of putting four blocks in the bag, for example, would require that she completely redo the graph.

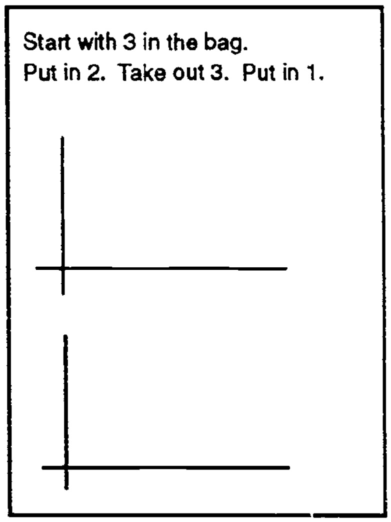
When, for the third time, Mark asks Alice to rethink the graph in order to show the number of blocks in the bags, she redraws the same graph but adds little squares to specify the number of blocks. Mark's reiteration of the question, namely, how to show the number of blocks, moves Alice to impose the required information of the number of blocks on a graphical scheme that is too centered on the changes.

In another paper (Tierney & Nemirovsky, 1991) we distinguish between system-driven representations and data-driven representations. A system-driven representation is determined by a consistent set of rules that create a realm of possibilities regardless of their existence. On the other hand, a data-driven representation is shaped by what is expected to happen or by what is deemed significant in actual events. Students create representations that have a mixture of system and data-driven elements with different levels of predominance. Alice develops a data-driven representation in which the changes, moreover the particular changes used in the original story, become the salient pieces of information that the graph has to show. Alice's information-driven approach is not necessarily "simpler" than a system-driven approach. Graph 4 shows how complex and consistent her graphical system is and how well she can keep the correspondence between the two graphical methods she has created (see Attachment 3, page 53).

Rose (age 9; 14-minute episode)

Segment 1: Graphing 3 +2 -3 +1

Mark shows Rose the sheet:



Mark: Now what I would like you to do, if you can think of a way to do it, is use this [the upper axis] to make a graph of that problem. Have you made graphs before?

Rose: Mmm-hmmm. [assenting]

Mark: Can you think of a way to show that problem and what happens in it, by making a graph?

Rose: Hmmm. I can think of one way which would be kind of hard, but I'll try.

Mark: What would that way be?

Rose: That would be to, um, have lots of, um, lines [gesturing vertical lines] and just have the spaces in between [pointing to different heights on the vertical axis] and you'd have one, two, three and so on up to the top. And then in every column you'd show what the changes were. You'd show like the numbers that were in the bag then.

Mark: Oh, I'd like to see that if you can do it.

Rose: [Rose draws 4 vertical lines and four horizontal ones, see Graph 1, page 29.] I don't know if this is going to be enough . . . [the number of horizontal lines] This part already has three in the bag [shades three blocks]. Put in, let's see, [adds another horizontal line] you've got five . . . put in two [shades five blocks].

Mark: What have you got there [in the second column]?

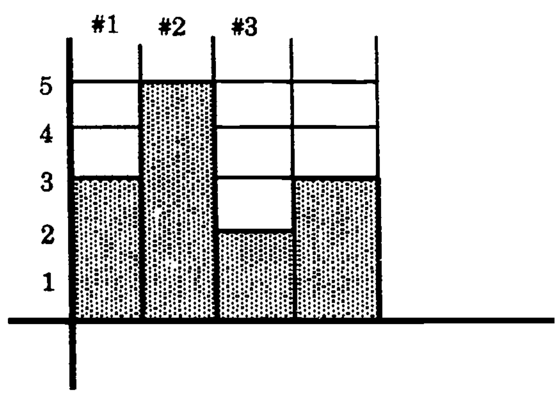
Rose: Five.

Mark: How did you know?

Rose: Because everybody knows that three and two is five. [tone of obviousness]

Mark: Oh, so it shows . . .

Rose: It shows what the answer is after the problem. I'll number them, number one, number two, number three [adds number on top of each column 1, 2, 3; each number corresponds to a change]. And you take out three and it would be two [shades two blocks]. And you put in one, and that would be three [shades three blocks].



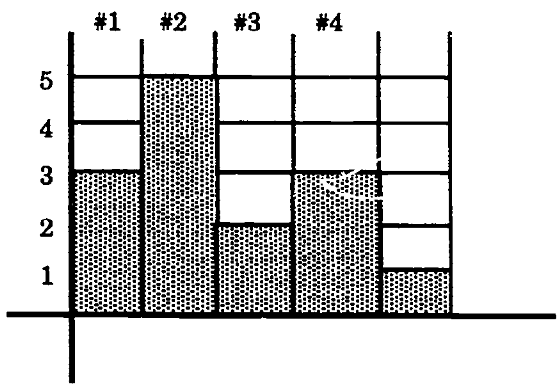
Graph 1

Analysis

Rose reveals a systemic way of thinking about the graph. She is not bounded to the particular story of changes posed in the sheet. First, she pictures columns with vertical spacings and, in trying to determine the semantics of those columns, she shifts from the changes to the number of blocks in the bag: "And then in every column you'd show what the changes were. You'd show like the numbers that were in the bag then." She strives to figure out a generic definition of what the columns would stand for, as opposed to using the columns to denote the literal sequence of events described in the story.

Segment 2

Rose explains the meaning of Graph 1 in different ways. She adds a fourth change specified by Mark (take out 2), obtaining Graph 2:



Graph 2

Mark: Would you say that that then is showing you what's in the bag at each time or the changes at each time? What would you say?

Rose: It could be showing you either of them, because it shows you what was changed. It doesn't really show you what was changed because you have to figure that out. But it shows you what was in the bag, what the changes were that made it different in the bag. So you'd have . . . um, you could tell that two is the change because you could level it out, like go all the way across the three line, and say, oh look what a difference [from 3 to 5]. And . . .

Mark: What do you mean by that?

Rose: Like you could go across the line that it was up to from the last one and see what happened, like if it's lower or higher than that line, and that's how much.

Rose illustrates her idea by moving her hand horizontally to show how one could figure out the change in going from five to two blocks in the bag.

Rose: You'd go across and you'd say, oh, there's nothing there. And you'd . . .

Mark: Take a drop.

Rose: And you'd fall down. And you'd say, ah, there we are. And you'd count the squares that you had dropped by.

Mark: Like an elevator.

Rose: Level four. Level three, level two [laughing].

Mark: Fortunately, you'd land safely on a two. All right, here's a question. This we might have to think about a little bit. Could you make a second graph that shows you what each change was?

Rose: What each change was. Hmmm . . .

Rose: I'm not sure, because I have no way of . . . I mean I could show the numbers each time, but I couldn't show whether it was plus or minus.

Mark: Why not?

Rose: Because I can't do that without showing what's in the bag at each time, I don't think. Because you'd still have to figure it out. Because I could do the numbers right, but you'd still have to figure it out a bit because you'd have to look to see whether it was higher or lower than the last time.

Mark: So you said you could do the numbers. What would the number be for this change?

Rose: The number would be two because you put in two [from 3 to 5].

Mark: What would the number be for this one [from 5 to 2]?

Rose: That would be three.

Mark: OK.

Rose: It would be like, let's see, whatever is not there, like three [indicating the three empty blocks from 5 to 2], and then there was . . . or whatever is there [pointing to the shaded block up from 2 to 3] from the last one.

Analysis

Rose discriminates between the information that is directly perceivable in a graph and the information that can be inferred from it: "It could be showing you either of them, because it shows you what was changed. It doesn't *really* show you what was changed because you have to figure that out." Trying to describe how the changes can be inferred from her graph, she articulates what one should look at and through which sequence. This sequence of perceptual and counting acts can be summarized as follows:

> Look at the initial number of blocks, take it as a referent line ("go across the line that it was up to from the last one").
>
> Notice whether the next level is higher or lower ("see what happened, like if it's lower or higher than that line").
>
> Count the number of blocks needed to reach the next level ("that's how much").

The distinction between going up or down permeates Rose's thinking through the whole segment. She talks about "falling down" and counting "the squares that you had *dropped* by." The main obstacle that she is trying to overcome, in devising a changes graph, is that she does not think of a natural way to show the up/down distinction. Rose makes this very clear when she says: "I could show the numbers each time, but I couldn't show whether it was plus or minus." Rose recognizes that her column-based graphical scheme is suitable to "do the numbers right but you'd still have to figure it out a bit because you'd have to look to see whether it was higher or lower than the last time." These utterances reflect how Rose, as well as all the children we interviewed, identifies an essential difference between changes and amount of blocks in the bag: a number suffices to indicate the amount of blocks, changes involve two pieces of information, a number and a more/less. Therefore, if a graph fails to show the more/less distinction, it does not reflect the changes.

Rose tells Mark about another way of perceiving and quantifying the more/less in her graph. Instead of the "elevator" counting steps to get to the new level, she suggests looking at "whatever is not there" (in the case of a decrease) or "whatever is there" (in the case of an increase). Now the critical features are not interpreted as dropping or raising levels but rather as emptiness or fullness. We have seen this way of picturing the changes in other children's interviews. Another child indicated a decrease by showing the blocks that had been taken out with white squares (as opposed to black squares for blocks that had been added); later, to emphasize this distinction, he added a skull to the white squares calling them "ghosts." This approach enabled the child to make pointable the blocks that were not in the bag, and to maintain a correspondence between each square in the graph and a particular block that was moved in and out of the bag.

Segment 3

Mark: Can you think of any way . . . You can invent a way if it comes to you, of showing those different kinds of changes. You'd have a hard time showing the minus ones, or whether it was the minus ones and the plus ones, but I'm just wondering if you can think up some way of doing that.

Rose: Well, I could . . . This is kind of stupid, but for the plus, I could start it from the bottom and for the minus I could come down from the top. [See Graph 3, page 33.] Then I'd have to do like next to it, one, two, three, four, five [numbers decreasing from the top, she writes them on the left side]. On the other side I could do it.

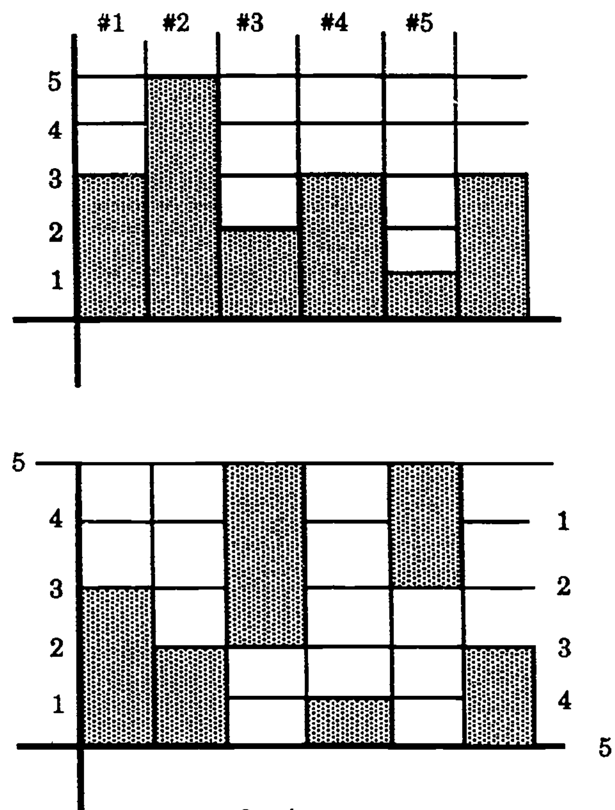
Mark: Oh, I see. So could you just give me an example, just draw me an example of what that would look like on this graph?

Rose: OK, [draws five vertical lines on the bottom graph and four horizontal ones] it's messy, but (. . .) um, then you'd probably have a one, two, three, four, five, [increasing up, on the left] and then you'd have a. . . one, two, three, four, five [increasing down on the right side].

Rose adds another line at the bottom corresponding to zero, so that she gets up to five vertical units. But later she dismisses this additional line and, instead, adds a fifth line on top of the graph.

Mark: So what have you shown so far?

Rose: So I've shown three, and then you put in two [shades two blocks, see Graph 3]. And then you take out three [shades three blocks counting down from the top]. Then you put in one [shades one block]. And you take out two [shades two blocks from the top]. That.



Graph 3

Mark: So, I see what you mean now. If it goes up from this line it's what you put in. If it comes down from here . . .

Rose: That's what you take out.

Mark adds a sixth column in the changes graph corresponding to +2, and Rose completes the graph of the number of blocks with another column indicating three blocks in the bag (see Attachment 4, page 54).

Analysis

In this segment, Rose constructs a graphical approach to show the changes. At the beginning she does not feel secure about her idea ("this is kind of stupid") to distinguish droppings and raises: "for the plus, I could start it from the bottom and for the minus I could come down from the top." She perceives the need of independent scales for increases and decreases respectively: "On the other side I could do it."

Her feeling of uneasiness may be related to her perception of a lack of systematism in the changes graph: the location of the "top" line (from which she would "come down") is too dependent on the specificity of the story (five was the maximum number of blocks in the bag) and the first column of three squares does not really belong there; it is not a change. Needing to include the initial number of blocks with the changes has more general implications. In some contexts, children tend to consider the initial amount of blocks as a change from zero. In other words, they have an inclination to think that the bag really starts empty and that the initial amount corresponds to the first change. More likely here Rose includes the initial "3" in the changes graph just because she wants to preserve this information. Otherwise, the number of blocks in the bag could not be "figured out" from the changes graph and something essential would be missed. In any case, Rose's approach shows a remarkable similarity to the conventional way of graphing negative changes "coming down" from the zero line.

Directions of Change

We investigated directions of change with Rose, who participated in the graphing exercise above, and with Carol, who had completed second grade. In the following two episodes, the students explore problems of maxima and minima; that is, questions about when the most and the least are reached in a sequence of changes. In the first part, the changes are specified with integers; in the second part, they are sequences of signs indicating successive increases and decreases.

Rose (age 9; 8-minute episode)

Segment 1: The Most and the Least

Mark: I will ask you to write some changes for yourself. First start with three... Say that this is what you started with in the bag. Then I put in six.

Rose: [Rose writes 3.] What if I do this arithmetic real fast? [Rose writes 9.]

Mark: Try to record them as changes before you do the arithmetic... Then put in one, then take out five, then take out three, and then put in two. Rose writes 3 +6 +1 -5 -3 +2.

Mark: I'm not going to tell you what you end up with.

Rose: OK.

Mark: What I want to ask you about—this is a slightly different question—when do you think I have the least in the bag? And this is more of a question to sort of give me a first impression versus to figure it all out.

Rose: Between minus three and plus two.

Mark: Why there?

Rose: Because you've taken away five and three... and then you spoil it by adding two.

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Mark: At this point [after the -3] I have less than I had at the beginning which is three?

Rose: Maybe you do, maybe you don't. I don't know, I think you might. I think so because, yes, six and one is seven and five and three is eight. So yes, you've got the least [after the -3].

Mark: . . . Where in the problem do you have the most?

Rose: Right there, after the 1.

Analysis

Mark's requests to record the changes before doing the "arithmetic" and to give a "first impression" seem critical because they move Rose from the computation of accumulated values to the focus on relations among the terms of the sequence. In doing so, she adopts a probabilistic language, such as: "Maybe you do, maybe you don't. I don't know, I think you might." Rose uses calculations to cope with uncertainty. This is an example of how children use the interplay between number relations and counting. She has the "impression" that the least has to be after the taking outs, which is a judgment based on number relations. But Mark's asking her to compare the point of the least amount with the initial amount prompts her to count the additions and subtractions.

This problem introduces a fundamental shift regarding the role of the order of changes. The problems that Rose executed earlier with net change, similar to the ones described with Chris and Jay, conveyed the idea that order is irrelevant (the net change was not affected by it), whereas in this new situation order is critical. This is reflected in the language that Mark and Rose use; it is full of temporal markers ("after," "you do this and then, . . . " "at this point," etc.).

Segment 2

In this segment, Mark asks Rose what she would need to do to have the most after the -5 in the same problem, 3 +6 +1 -5 -3 +2.

Mark: Suppose I wanted the most to be right here? [pointing to after the -5] What would you have to do to the problem to make that the place where there was the most in the bag?

Rose: Well you could change the first number . . . You could change it to something a lot higher, like ten maybe. You could change the sign there [on the -5] to a plus.

Mark: Is there any other thing you could do?

Rose: You could change the 6 or the 1 to something really high.

At this point Rose changes her mind.

Rose: But I don't think that would work. The reason I don't think you could change any of these [pointing to 3, +6, +1] to something higher is because you'd still

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have to take away the 5 and then you'd have less ... Right after the one you would still have more. I think you would have to change the minus five into a plus five.

Analysis

This segment shows Rose experiencing a fundamental insight; namely, she distinguishes between how high is a number and how it differs by sign from its neighboring numbers in the sequence. At the beginning of the segment Rose tries to find ways of "moving up" the number accumulated before the -5. But she realizes that the increase in the previous accumulations will also move the neighboring values "really high," so that the most *cannot* be after -5. This learning relates to central ideas of the mathematics of change:

> A change of a quantity-change generates a uniform translation of *all* the subsequent quantities.
>
> A local extreme is determined by local behavior; that is, the change in sign between two contiguous changes is sufficient to make the intermediate accumulated quantity a local extreme.

Segment 3

Here Mark poses questions about most/least given the sequence 4 + + + - - +, so that only the directions of change are known.

Mark: I'm going to show you some changes but I'm not even going to tell you what they are exactly. Let's just say you start with four ... I'm just going to show you the signs, [writes 4+++ - - +] equal some number at the end, but let's not worry about the number at the end for now.

Rose: What do I get to do, put in the numbers?

Mark: Well, not first off. First off I'd like you to tell me, if you can, where I would have the most in the problem?

Rose: Right there [after the +++], because you'd just had a lot of pluses and then you'd have to go through the minuses to get back to plus. I think you'd have the most right there [after the +++].

Mark: What does going through pluses have to ...

Rose: Adds more and more and more. Because there's three pluses—more, more, more [hand gesture toward all three +++]. And then when you go through the minuses [brief hand gesture toward the - -], there's less and less and then when you go back to the plus there's more, but you've already taken out two of these guys [bracketing two pluses]. Say it's all ones. OK? I'm just trying to give an example. There's one all the way across.

Mark: By one, you mean each change is a one?

Rose: I mean every number is a one. But anyway, you take away two of the ones [with the two minuses], and then you add one more. That's only six, but here [bracketing the 4+++] you've got seven.

Mark: So if they are all ones, it works out like you said. That's where you've got the most.

Rose: Yeah.

Mark: Is there any situation where it wouldn't work out that way?

Rose answers, "I don't think so. . .unless. . . ," and she tries to figure out changes on the sequence of signs that would move the most to a different place. But Mark reacts with a more specific question.

Mark: Is there anything about this plus [last +] that could make you change your mind?

Rose: Only one thing, if that was a lot . . .

Mark: If the last one was a lot . . .

Rose: Yeah, like ten or so [chuckling]—then probably that would be the most provided these [points briefly to +++, then more firmly to - -], provided these [- -] are not too much and these [+++] are not too much either.

Mark: So in other words you have to have a real big one to end up with the most at the end?

Rose: Yup.

Analysis

Rose starts off interpreting each plus or minus as showing an equal-size increase and decrease, a "more" is neutralized by a "less." She makes her idea very clear with an example: "Say it's all ones." An important shift occurs when Marks points toward the last +, Rose immediately realizes that there are other possibilities: "if that was a lot. . . ." This shift moves her to notice and think on the basis of the *size* of the changes. The notion of order of magnitude comes to be significant: "like ten or so." The language of relative size becomes dominant in the conversation: "a lot," "too much," "real big," etc.

In her final analysis about the conditions for the most to be at the end, Rose shows a difficulty organizing what to pay attention to. She deems that the most will be at the end, "provided that these [- -] are not too much and these [+++] are not too much either." She does not realize that the only aspect that matters is for the last plus to be bigger than the two minuses together. The size of the previous pluses is in fact irrelevant. Rose has a sequence with two maxima (after the four pluses and after the last plus) and wants to make sure that the latter would be bigger than the former one by lowering the four pluses, without noticing that this lowers both maxima together.

Carol (age 8; 17-minute episode)

Segment 1: The Most and the Least

In this segment, Mark asks for the places of the highest and the lowest in the bag, given the sequence 3 +6 +1 -5 -3 +2.

Mark: Say this was a problem where we started with three and then we did these changes one after the other. At what point, after which change would I have had the most in the bag?

Carol: Plus six.

Mark: Why there?

Carol: Because that's the most you're ever going to put on.

Mark: So you think right here [after +6] is where I have the most in the bag?

Carol: Yeah.

Mark: How about here [after +1], would there be more or less than I had here [after the +6]?

Carol: More.

Mark: But how can this [after +6] be the most and then that [after +1] be more?

Carol: Oops. This would be the most right here [after +1] . . .

Mark: Why? Why would that one [after +1] be more than that [after +6]?

Carol: Because you add on one more number.

Mark: OK. Where would I have the least, do you think?

Carol: Minus five [pointing after -5].

Mark: Why?

Carol: Because that's when you minus the most. No [with emphasis], the minus three . . .

Mark: How about at this point where there's a plus two [at the end]? Do you think it had the most or the least or somewhere in between or?

Carol: Somewhere in between.

Mark: Why do you say that?

Carol: Because you've added on these [+6 +1], but then you just minus them [-5 -3], so . . .

Mark: Do you think I could have more here [at the end] than I did at the beginning?

Carol: Because you didn't minus all you had here from here [+6 +1] to get to here [after -5 -3], then you would add more here [+2] . . . So this is probably not going to be one right here [before +2]. And if it's one then that [the result at the end] will equal three. But if it's not one then it won't equal three [the starting number].

Mark: You think it's probably not going to equal one here [before +2].

Carol: Yeah.

Mark: Why do you think it won't?

Carol: Because you didn't minus everything you had but one . . . Should I do it now?

Carol writes 3 +6 9 +1 10 -5 5 -3 2 +2 = 4 and verifies that the least is where she has predicted it to be and that there is more at the end (4) than at the beginning (3), see Attachment 5, page 55.

Analysis

In this segment, Carol moves from noticing the extremes in the quantity-changes to focusing on the extremes in the quantities. Mark creates a critical experience by asking her to compare the amounts after +6 and after +6+1. Note that her initial reaction is not a matter of "confusing" quantities with quantity-changes. Her language is explicit from the beginning: +6 is "the most you are ever going to put in there" and -5 is "when you minused the most." Initially, she clearly interprets the "most" and the "least" as the biggest change.

The problem-situation leads Mark and Carol to adopt a language of probabilities and relative certainty: "somewhere in between," "I think so," "probably," etc. These judgments are to be checked with the results of actual computations. Carol refers to the counting of the intermediate numbers of blocks in the bag as "doing" the problem. Carol uses counting as a testing procedure. Knowledge of number relations on the one hand, and knowledge of counting and computing on the other hand, configure two domains that both relate to each other and offer distinctive perspectives.

Segment 2

Mark poses a problem similar to the one introduced in Segment 1 but with different numbers. Then Mark asks Carol to think about this sequence of changes + + + + - - - - , in which only the directions of change are specified.

Mark: [writes +++ - - - -] Now, you notice I haven't shown you how big each change is, just what I either put in or take out. Now I want to ask you if you can tell me where the most would be in the bag?

Carol: Right here [after the last plus].

Mark: And that's because why?

Carol: Because you've added everything before then, you haven't taken away anything.

Mark: And where is the least going to be?

Carol taps the last minus sign with her finger.

Mark: How do you know?

Carol: Because you've taken away lots of things.

Mark: Because you've done all taking away?

Carol: Yeah.

Mark: OK, does it matter what the numbers are? Where the biggest or the least would be? The most or the least?

Carol: Mmm-hmm.

Mark: How would it matter?

Carol: Because if you did a big enough, if the answer to all of these [++++] would be like twenty or something, and the answer to minus all of these [- - - -] would only be like ten, and then you'd still have a lot here [at the end].

Mark: . . . could you have the smallest number at the very start?

Carol: Yeah.

Mark: How would that work?

Carol: If you did big numbers here [++++] and small numbers here [- - - -].

Mark asks what would happen if all the changes were one. Carol counts and finds, expressing some surprise, that it would end with the same number as the beginning.

Mark: Do you think it would still happen if I changed one of these [++++]? Made one of these pluses bigger? If I made one of those pluses bigger then what do you think I'd end up with?

Carol: It depends on what number.

Mark: Would it be bigger or smaller than I started with?

Carol: It would have to be bigger.

Mark: Why?

Carol: Because you probably wouldn't do a plus zero.

Mark: No, probably not.

Carol: So it would be more.

Mark: . . . All right, here's a new wrinkle. Suppose this was what all the changes looked like [adds a + at the end, creating ++++ - - - - +]. We added one change at the end and it was one of those. A put in, a plus change.

Carol: Now what would the numbers be?

Mark: Well, let's say we don't know what the numbers are. Would you be able to tell me where the most would be?

Carol: It depends. It depends on how much you take away from here [- - - -] and how much you add here [++++].

Mark: So give me some examples of what it might be like.

Carol: So if you did really big numbers here [++++], and then small or medium numbers here [- - - -], then you did this one [the last plus], probably no matter what number it [the last plus] is, this [after the ++++] would probably be the biggest one.

Mark: Right there [after the ++++]?

Carol: Yeah. But if you did like small numbers here [+++] and a little bigger here [- - - -], then this might be the biggest one [after the last +].

Analysis

Carol expresses in this segment an important distinction between situations in which the most/least has to be at a certain place, regardless of the size of the changes, such as the most in ++++ - - - -, and situations in which the size of the changes matters, such as the least in ++++ - - - - or the most in ++++ - - - - +. For example, she judges that the location of the least depends on "what the numbers are," and then she constructs an example. Later, with another question, she says "it depends." When the size of the changes are relevant, she spontaneously uses orders of magnitude as a way of discriminating the possible cases: "is like twenty or something," "would be like ten," "big numbers . . . small numbers."

Her sense of logical necessity regarding relations of more/less is clearly expressed. For example, she says that if all the changes are one, except for one of the pluses, the ending number "would *have* to be bigger." She recognizes that the only possibility for this not to be true is with the exceptional plus as zero, so she says "Because you probably wouldn't do a plus zero."

As Carol tries to figure out the conditions for the most in ++++ - - - - +, she makes the same mistake that Rose makes in Segment 3. She thinks that big numbers in ++++ and small or medium in - - - - will set the most after ++++, regardless of the value of the last +, whereas small numbers in ++++ will move the most to the end. She appears to be thinking of how to make the number of blocks after the four pluses high or low, without recognizing that this up or down equally drags the number of blocks at the end up or down.

Segment 3

After exploring more about the possibilities of having the most/least at the beginning or at the end, Mark asks:

Mark: Have you ever done something called a time line?

Carol: No.

Mark: It's a way of taking a line and showing what happens between two different times. So say this was at the start of this bag problem, OK, and this is the end of it. This bag problem right here, we just see the pluses and minuses:

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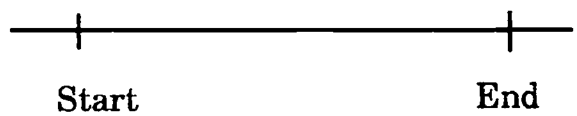
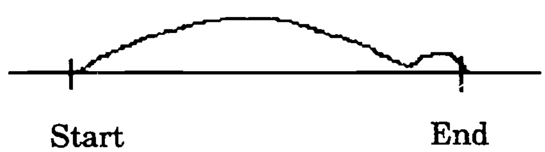
+ + + + - - - - +



Mark: Now suppose there's a situation in which these [++++] are really big, these [- - - -] are pretty small, and this [the last plus] is kind of medium. Can you draw a line going from here [start] to here [end], it could be like a curvy line or any kind of shape that shows how many are in the bag at each point as you go from here to here [along the sequence of changes]?

Carol draws a curved line:

+ + + + - - - - +



Graph 1

Mark: Now that's an interesting shape. Could you explain it to me?

Carol: Well, here [the increasing portion of the first bump] it's getting higher and higher and higher. And when you get to here [the highest point, also pointing to the change from the last plus to the first minus] it starts getting lower and lower and lower and then it goes up a little more.

Mark: It goes up a little?

Carol: Yeah.

Mark: And why did it go down at the very end?

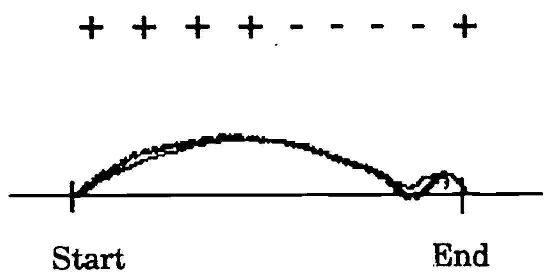
Carol: Because it's easier just to end it right here [on the horizontal line], you don't really have [to go down at the end]. It would just like end up here [above the horizontal line].

Mark: Oh, I see, the real end would be right here at this little top [the highest point of the second bump].

Carol: Yeah. But it's easier just to make it go down like that.

Mark: I see. How would that change . . . this is interesting. I'll give you a different color. Suppose the start numbers were pretty big [++++] and these [- - - -] numbers were about the same size . . . the minuses were about the same size as the pluses. Then what would it look like? And this [the last plus] was in the middle . . .

Carol: It still goes like this [the increasing part of the first bump], but then it goes like that [all the way down to the horizontal line]:



Graph 2

Mark: You end up where? Down lower? [after the down part, touching the horizontal line]

Carol: And then you go up here [top of the little bump].

Mark: I see. So the only difference is it seemed like you went lower here [when the new line touches the horizontal line].

Carol: Yeah.

Mark: If I made one that went like this, sort of was the same as this [the up part], but then it stayed almost as high.

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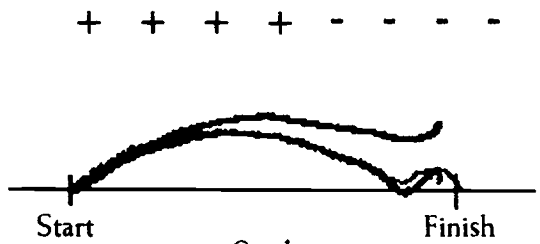
Mark draws a third line in another color:



Graph 3

Mark:	What do you think, what would that tell you about the changes that I actually made?
Carol:	. . . Those [++++] were big. That [- - - -] was little and that [the last +] was big.
Mark:	OK, and how did you know, or what made you think that these changes [- - - -] were small?
Carol:	Because this [the down part of the graph] doesn't go down very far.
Mark:	So the bigger the change [pointing to the minuses] the more it's [the curve] going to go down.
Carol:	Yeah.

Analysis

As Carol strives to create a continuous description, she maps the pluses with increases, the minuses with decreases, and indicates that the maximum corresponds to the change of sign. Her language also suggests continuous variation: "it's getting higher and higher and higher" and "it starts getting lower and lower and lower."

As Carol elaborates, she makes clear that she is ending her curve on the horizontal line, because there she finds the label "end," but she knows that "you don't really have that," it is just easier "to make it go down."

The new curve in her Graph 2 has two differences with respect to the curved depicted in Graph 1:

> After the first bump it goes down to the initial height on the horizontal axis to show that the minuses were "about the same size" as the pluses.
>
> It ends at the top of the second bump showing that the final number of blocks will be greater than the initial one.

Carol's interpretation of the third curve drawn by Mark shows that she recognizes the size of the changes in how much the curve goes up or down. Using Mark's idea of drawing a continuous line to depict the discrete changes of quantity, Carol is able to express qualitatively the accumulation resulting from the changes (see Attachment 5, page 55).

Section 5: Discussion

We do not make any particular claim about how representative the children's ideas are, as manifested in the interviews. Rather, we gain a sense of the range of students' approaches to additive change. Inspired by this exploratory study, we have developed curricular materials that are being piloted in three elementary schools in the Boston area.

We see three main themes emerging from the selected episodes:

Additive change: a web of relationships. The contrast between accumulations and changes is salient for children dealing with situations of ins and outs of blocks in a bag. Children are able and careful to go back and forth between the situations of ins and outs without losing sense of the respective identities. Remember Chris saying "there's one and then two and then three" and writing 1+1+1. To distinguish changes, children need a more/less marker to be specified. We saw Chris, for example, inventing notational marks, such as "p" and "t," and Rose struggling to create a graphical system for the changes in which the more/less distinction would be shown.

Children do not articulate general principles on when and how the order of the changes affect the results, but they are sensitive to issues of order. For example, in trying to solve problems of maxima and minima for the accumulated values whose location is affected by the order of the changes, we recognized how Rose and Carol adopted a language full of temporal markers ("after," "do this and then," "at this point," etc.). This observation emphasizes the relevance of working with sequences. Otherwise, the richness of relations of order do not arise.

The interplay between number relations and counting, as two related but independent perspectives, is a critical aspect of children's approaches to additive change. Jay, for instance, transformed his counting of the changes (+1 +4 -3) to construct the number relation two-more: "It would be two more in that [bag]"; later, he used the two-more relation to discriminate what is possible and impossible in the situation. Often a problem can be thought of in both ways, focusing on counting/ computing and on number relations. When Mark posed the problem of the least in 3 +6 +1 -5 -3 +2 to Rose, she started to compute, but Mark's request to give her "first impression" was enough to move her into an analysis of number relations.

Composition and decomposition of small numbers are critical in children's approaches to additive change. For example, Chris, noticing that he needed one more change to go from 6 to 4, split a -4 into -2 -2. Children preserve conservation of net change by composing and decomposing changes.

The many graphs of additive change. Depending on the situation, children can take either the changes or the accumulated values as the primary information to be graphed. Alice focused on the changes and Rose on the accumulated values. Children differ markedly in their concerns for systematicity. Some children develop a more data-driven representation, such as in Alice's episode in which the graph was a way of transcribing a given story literally. Others, as in Rose's episode, want to articulate a method to graph any story of the kind with its own internal consistency.

Children clearly distinguish between what is directly perceivable in a graph and what is there but "you have to figure it out." This perception enables them to understand the difference between a graph of the number of blocks and a graph of the changes. Rose, as well as all the children who succeeded in creating two graphs during the interviews (one for the accumulations and another for the changes), showed fluency in going back and forth between them.

Carol exemplified children's ability to use continuous descriptions for discrete change. She used a continuous line to express her judgments of "more" and "less" in terms of "higher" and "lower." Her language indicated how she was experiencing the discrete changes as continuous and gradual: *"it's getting* higher and higher and higher" and *"it starts getting* lower and lower and lower."

Qualitative relations. When Mark raised problems involving qualitative judgments, such as the most and the least in a sequence or sequences of pluses and minuses, the children spontaneously started to think about orders of magnitude. For example, Carol hypothesized "is like twenty or something," or Rose "like ten or so." "Not too much," "a lot," "real big" became descriptors that helped them to figure out the different possibilities. By using orders of magnitude the children explored when and how the size of the changes matters.

Carol's shift from looking for the most/least in the changes to the most/least in the accumulations, as well as Rose's realization that an increase in a change increases all the successive accumulations, made apparent how deeply involved qualitative relations were in her understanding of additive change. For example, Carol said that -5 was when "you minused the most," but immediately she recognized that this was not the same as having the least in the bag.

These episodes have helped us to understand that qualitative reasoning is not pre-quantitative or less sophisticated than quantitative judgments. Focusing on number relations and judgments of more/less/same brings up a distinctive perspective for handling additive change, which is different from the perspective elicited by counting and computing. Qualitative thinking enables us to make sense of numeric results and to have expectations as to what is possible or impossible. It grows in parallel with quantitative knowledge and, pursuant to our interests in formulating a long-term view of calculus learning, it is one of the domains that we intend to study further in the near future.

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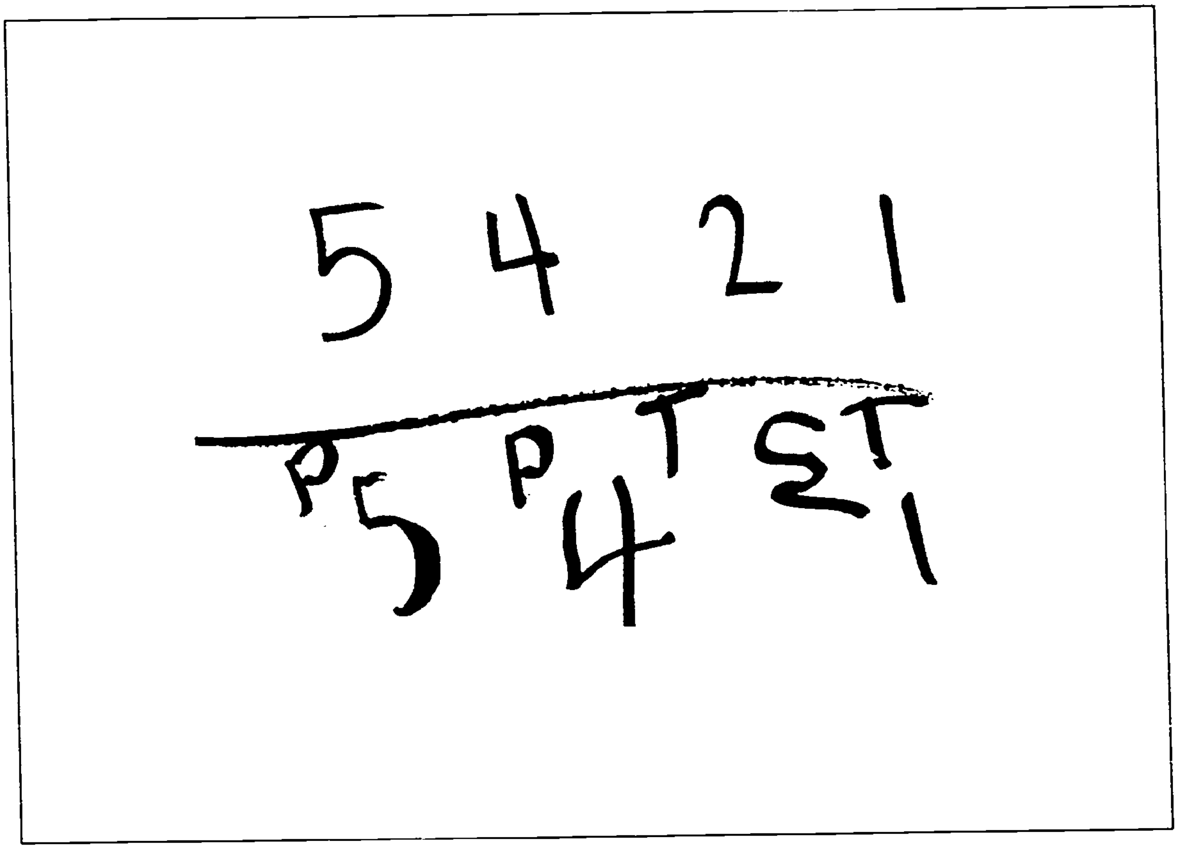
Attachment 1



Attachment 2

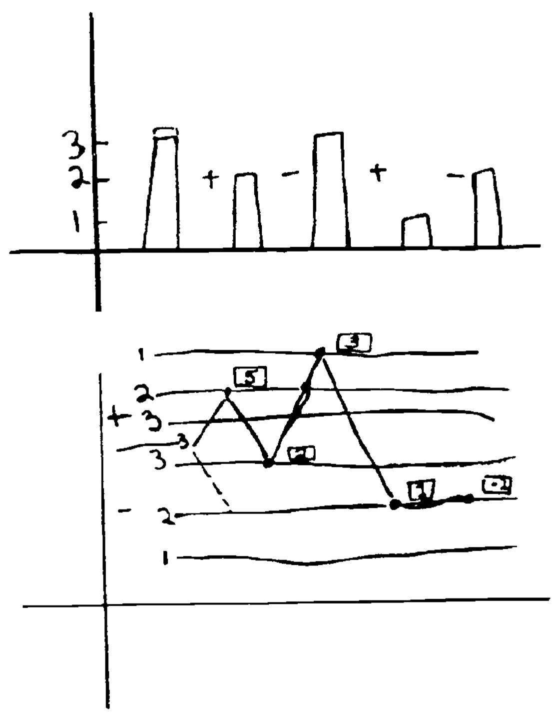
$$4 - 14 - 1 + 2 + 3$$

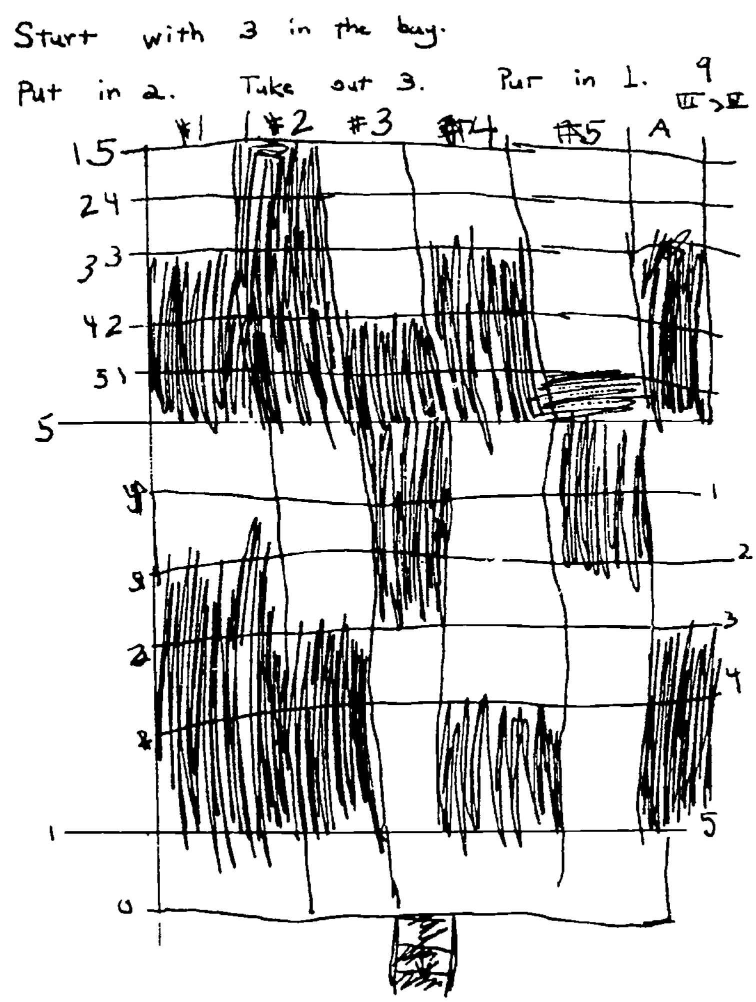
$$4 - 14 - 1 + 2 + 2$$

Attachment 3

Start with 3 in the bag.

Put in 2. Take out 3. Put in 1.




Start with 3 in the bay.
Put in 2. Take out 3. Put in 1. 9
#1 #2 #3 #4 #5 A
15
24
33
42
51
5
1
2
3
4
5
1
0

3 + 69+ 10- 55 32+ 2 = 4

5 + 2 + 3 + 1 - 2 - 1 - 2 - 4

+ + + + - - - - +

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